1.5 Solution sets of linear systems

Wednesday, January 30, 2019 1:05 PM

"Now that we have fully connected linear systems with equations involving vectors and matrices, we use our geometric and algebraic understanding of vectors to gain similar understanding of solution sets of linear systems. To begin, a special class of linear systems...."

Homogeneous linear systems

"systems with all zero right hand sides"

A linear system is said to be homogeneous if it can be written as a matrix equation $A\vec{x} = \vec{0}$.

Note: $\vec{x} = \vec{0}$ is always a solution to such a system: $A(\vec{0}) = \vec{0}$.

For this reason we call it the trivial system and concern ourselves with nontrivial solutions, i.e. $\vec{x} \neq \vec{0}$ s.t. $A\vec{x} = \vec{0}$.

Ex) Consider the three homogeneous systems:

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_1 - 3x_2 - 2x_3 &= 0 \\ -x_1 + 3x_2 + 2x_3 &= 0 \\ 2x_1 - 6x_2 - 4x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_1 + 5x_2 - 3x_3 &= 0 \\ -3x_2 + 2x_3 &= 0 \\ 3x_1 - 3x_2 + x_3 &= 0 \end{aligned}$$

In matrix form

$$A_1\vec{x} = 0 \qquad A_2\vec{x} = 0 \qquad A_3\vec{x} = 0$$

where

$$A_1 = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & -3 & -2 \\ -1 & 3 & 2 \\ 2 & -6 & -4 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 5 & -3 \\ & -3 & 2 \\ 3 & -3 & 1 \end{bmatrix}$$

To solve these homogeneous systems we row reduce

$$\begin{bmatrix} A_1 & \vec{0} \end{bmatrix} \qquad \begin{bmatrix} A_2 & \vec{0} \end{bmatrix} \qquad \begin{bmatrix} A_3 & \vec{0} \end{bmatrix}$$

$$\| \qquad\qquad \| \qquad\qquad \|$$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & -3 & -2 & 0 \\ -1 & 3 & 2 & 0 \\ 2 & -6 & -4 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 5 & -3 & 0 \\ & -3 & 2 & 0 \\ 3 & -3 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 & 3 & 2 & 0 \\ 2 & -6 & -4 & 0 \end{bmatrix} \qquad \begin{bmatrix} & -3 & 2 & 0 \\ 3 & -3 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \sim \begin{bmatrix} 1 & -3 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \sim \begin{bmatrix} 1 & 0 & 6 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Solution sets: $x_1 = 4/3\, x_3$, $x_2 = 0$, x_3 free $\qquad x_1 = 3x_2 + 2x_3$, x_2, x_3 free $\qquad x_1 = 0$, $x_2 = 0$, $x_3 = 0$

In vector form: $\vec{x} = \begin{bmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{bmatrix}$ for any x_3 $\qquad \vec{x} = \begin{bmatrix} 3x_2 + 2x_3 \\ 0 \\ 0 \end{bmatrix}$ for any x_2, x_3 $\qquad \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Notice: $A\vec{x} = \vec{0}$ has a nontrivial solution if and only if there is a free variable.

We may describe the solution sets of $A_1\vec{x} = \vec{0}$ and $A_2\vec{x} = \vec{0}$ by way of parametric vector equations

$A_1\vec{x} = \vec{0}$ has $\vec{x}$ as a solution if $\vec{x} = x_3\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$ for any x_3

$A_2\vec{x} = \vec{0}$ has $\vec{x}$ as a solution if $\vec{x} = x_2\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ for any x_2, x_3

Notice x_2, x_3 are acting as parameters and any solution is given by a vector equation involving those parameters.

Equivalently, if $\vec{u} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and r, s, t are any real numbers

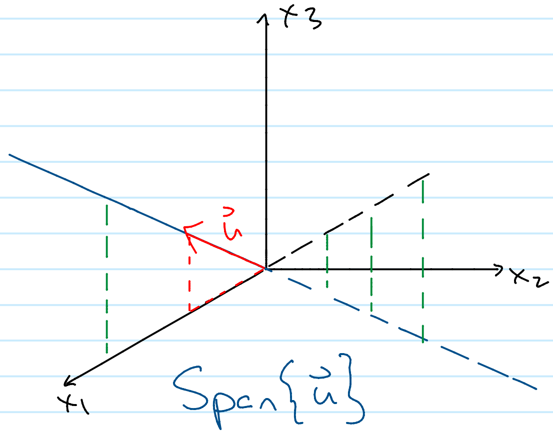
the solution set of $A_1\vec{x} = \vec{0}$ is all vectors of the form $\vec{x} = r\vec{u}$; and the solution set of $A_2\vec{x} = \vec{0}$ is all vectors of the form $\vec{x} = s\vec{v} + t\vec{w}$

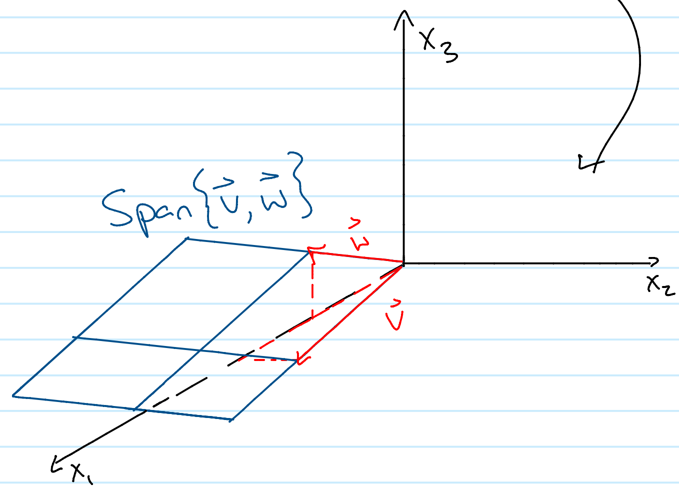
what is this set in $\mathbb{R}^3$?

$\uparrow x_2$

$x = ru$ $\qquad\qquad$ $\vec{x} = s u + t w$

what is this set in $\mathbb{R}^3$?





We call $\vec{x} = r\vec{u}$ and $\vec{x} = s\vec{v} + t\vec{w}$ parametric vector forms of a line and plane respectively.

"This is all well and good but how does this help us understand nonhomogeneous systems?"

Nonhomogeneous systems

"Most often in applications we are solving nonhomogeneous systems $A\vec{x} = \vec{b}$ ($\vec{b} \neq \vec{0}$). But in fact, as we will soon see, understanding the associated homogeneous system $A\vec{x} = \vec{0}$ will make describing the solution set of $A\vec{x} = b$ relatively easy."

Ex) Describe the solution set of $A_1\vec{x} = \vec{b}$ where $\vec{b} = \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix}$.

To find it, we row reduce:

$$[A_1 \ \vec{b}] = \begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & 8 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

notice row reducing A_1 looks no different $\qquad$ only this is new from homogeneous case

We see the solutions are

$$x_1 = -1 + \tfrac{4}{3}x_3 \quad \text{with } x_3 \text{ free.}$$
$$x_2 = 2,$$

In vector form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ \underset{\text{any value}}{x_3} \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\vec{p}} + x_3 \underbrace{\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}}_{\vec{u}}$$

(from before!)

Note! If $x_3 = 0$, $\vec{p}$ is a solution to $A\vec{x} = \vec{b}$.

Note: If $x_3 = 0$, $\vec{p}$ is a solution to $A\vec{x} = \vec{b}$.

We call $\vec{p}$ a particular solution to $A\vec{x} = \vec{b}$

and $\vec{u}$ a homogeneous solution for $A\vec{x} = \vec{0}$

(since $A\vec{u} = \vec{0}$, i.e. $\vec{u}$ solves the homogeneous system)

This is true in general ?

Fact: If the equation $A\vec{x} = \vec{b}$ is consistent then the solution set is all vectors of the form

$$\vec{y} = \vec{p} + \vec{v}_h$$

where $\vec{p}$ is a particular solution of $A\vec{x} = \vec{b}$

and $\vec{v}_h$ is any homogeneous solution (to $A\vec{x} = \vec{0}$.)

Ex/ Check! (reader)

$\vec{p} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ is a particular solution of $A_1\vec{x} = \underbrace{\begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix}}_{\vec{b}_1}$

$\vec{q} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution of $A_2\vec{x} = \underbrace{\begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}}_{\vec{b}_2}$

This implies that the solution set of:

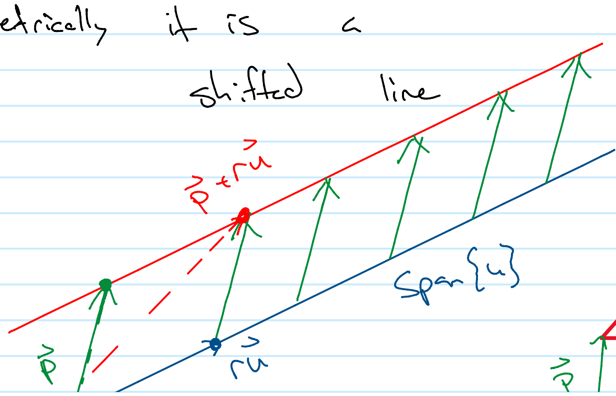
$A_1\vec{x} = \vec{b}_1$ is all vectors of the form $\vec{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + r\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$

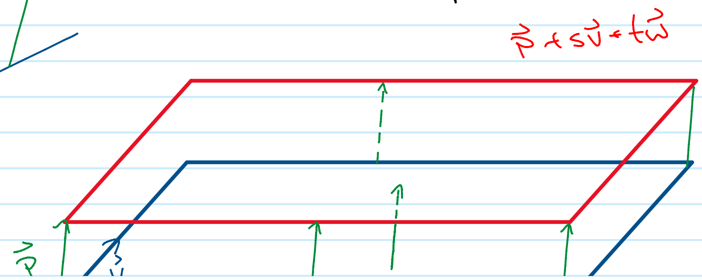
or $\vec{x} = \vec{p} + r\vec{u}$

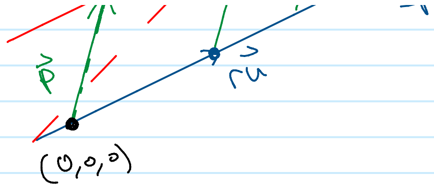
for $A_2\vec{x} = \vec{b}_2$ it is the vectors $\vec{x} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

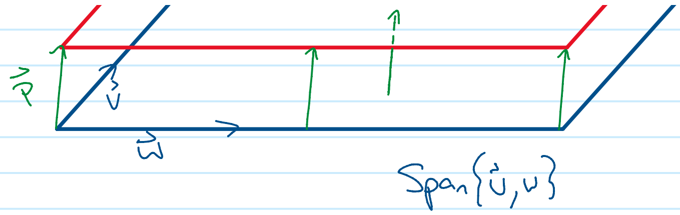
or $\vec{x} = \vec{q} + s\vec{v} + t\vec{w}$

Geometrically it is a shifted line and a shifted plane









These are given exactly on the webpage: notice the case with A_3, the solution set of $A_3\vec x=\vec 0$ is Span$\{\vec 0\}=\{\vec 0\}$.

The particular solution to $A_3\vec x=\vec b_1$ is $\begin{bmatrix}3.5\\4\\5.5\end{bmatrix}=\begin{bmatrix}3.5\\4\\5.5\end{bmatrix}+t\begin{bmatrix}0\\0\\0\end{bmatrix}$

and to $A_3\vec x=\vec b_2$ is $\begin{bmatrix}-1/3\\8/3\\5\end{bmatrix}=\begin{bmatrix}-1/3\\8/3\\5\end{bmatrix}+t\begin{bmatrix}0\\0\\0\end{bmatrix}$

so similar behavior is occuring, it is just degenerative.

Also, it is easy to verify that $\vec p+\vec v_h$ is always a solution to $A\vec x=\vec b$: $A(\vec p+\vec v_h)=A\vec p+A\vec v_h=\vec b+\vec 0$. ✓